

SOME INEQUALITIES OF MATRIX POWER AND KARCHER MEANS FOR POSITIVE LINEAR MAPS

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ABSTRACT. In this paper, we generalize some matrix inequalities involving matrix power and Karcher means of positive definite matrices. Among other inequalities, it is shown that if $\mathbb{A} = (A_1, \dots, A_n)$ is a n -tuple of positive definite matrices such that $0 < m \leq A_i \leq M$ ($i = 1, \dots, n$) for some scalars $m < M$ and $\omega = (w_1, \dots, w_n)$ is a weight vector with $w_i \geq 0$ and $\sum_{i=1}^n w_i = 1$, then

$$\Phi^p \left(\sum_{i=1}^n w_i A_i \right) \leq \alpha^p \Phi^p(P_t(\omega; \mathbb{A}))$$

and

$$\Phi^p \left(\sum_{i=1}^n w_i A_i \right) \leq \alpha^p \Phi^p(\Lambda(\omega; \mathbb{A})),$$

where $p > 0$, $\alpha = \max \left\{ \frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{1}{p}} Mm} \right\}$, Φ is a positive unital linear map and $t \in [-1, 1] \setminus \{0\}$.

1. INTRODUCTION AND PRELIMINARIES

Let \mathcal{M}_n be the C^* -algebra of all $n \times n$ complex matrices and $\langle \cdot, \cdot \rangle$ be the standard scalar product in \mathbb{C}^n with the identity I . For Hermitian matrices $A, B \in \mathcal{M}_n$, we write $A \geq 0$ if A is positive semidefinite, $A > 0$ if A is positive definite, and $A \geq B$ if $A - B \geq 0$. If m, M be real scalars, then we mean $m \leq A \leq M$ that $mI \leq A \leq MI$.

The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$ -isomorphism between the C^* -algebra $C(\text{sp}(A))$ of continuous functions on the spectrum $\text{sp}(A)$ of a Hermitian matrix A and the C^* -algebra generated by A and I . If $f, g \in C(\text{sp}(A))$, then $f(t) \geq g(t)$ ($t \in \text{sp}(A)$) implies that $f(A) \geq g(A)$. A linear map Φ on \mathcal{M}_n is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. A norm $||| \cdot |||$ on \mathcal{M}_n is said to be unitarily invariant norm if $|||UAV||| = |||A|||$, for all unitary matrices U and V .

Let $A, B \in \mathcal{M}_n$ be two positive definite and $t \in [0, 1]$. The operator t -weighted arithmetic, geometric, and harmonic means of A, B are defined by $A \nabla_t B = (1 - t)A +$

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tB , $A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ and $A!_t B = ((1-t)A^{-1} + tB^{-1})^{-1}$ respectively, in which $A!_t B \leq A\sharp_t B \leq A\nabla_t B$. In particular, for $t = \frac{1}{2}$ we get the operator arithmetic mean ∇ , the geometric mean \sharp and the harmonic mean $!$. The AM-GM inequality reads

$$\frac{A+B}{2} \geq A\sharp B. \quad (1.1)$$

In [12], Lim and Palfia have introduced matrix power means of positive definite matrices of some fixed dimension. If $\mathbb{A} = (A_1, \dots, A_n)$ is a n -tuple of positive definite matrices A_i ($i = 1, \dots, n$) and $\omega = (w_1, \dots, w_n)$ is a positive probability weight vector where $w_i \geq 0$ ($i = 1, \dots, n$) and $\sum_{i=1}^n w_i = 1$, then the matrix power means $P_t(\omega; \mathbb{A})$ is defined to be the unique positive definite solution of the non-linear equation:

$$X = \sum_{i=1}^n w_i (X\sharp_t A_i), \quad t \in (0, 1]$$

For $t \in [-1, 0)$, it is defined by $P_t(\omega; \mathbb{A}) = P_{-t}(\omega; \mathbb{A}^{-1})^{-1}$, where $\mathbb{A}^{-1} = (A_1^{-1}, \dots, A_n^{-1})$. We denote $P_1(\omega; \mathbb{A}) = \sum_{i=1}^n w_i A_i$ and $P_{-1}(\omega; \mathbb{A}) = (\sum_{i=1}^n w_i A_i^{-1})^{-1}$, the weighted arithmetic and harmonic means of A_1, \dots, A_n , respectively.

There is one of important properties of matrix power means $P_t(\omega; \mathbb{A})$, that $P_t(\omega; \mathbb{A})$ interpolates between the weight harmonic and arithmetic means:

$$\left(\sum_{i=1}^n w_i A_i^{-1} \right)^{-1} \leq P_t(\omega; \mathbb{A}) \leq \sum_{i=1}^n w_i A_i \quad (1.2)$$

for all $t \in [-1, 1] \setminus \{0\}$.

The Karcher means of n positive probability vectors in \mathbb{R}^n convexity spanned by the unit coordinate vectors, is defined as the unique positive definite solution of the equation:

$$\sum_{i=1}^n w_i \log \left(X^{\frac{1}{2}} A_i^{-1} X^{\frac{1}{2}} \right) = 0. \quad (1.3)$$

The Karcher means denoted by $\Lambda(\omega; \mathbb{A})$, where it follows from (1.3) that $\Lambda(\omega; \mathbb{A}^{-1})^{-1} = \Lambda(\omega; \mathbb{A})$. It is well known that (see [12])

$$\lim_{t \rightarrow 0} P_t(\omega; \mathbb{A}) = \Lambda(\omega; \mathbb{A}) \quad (1.4)$$

and

$$\left(\sum_{i=1}^n w_i A_i^{-1} \right)^{-1} \leq \Lambda(\omega; \mathbb{A}) \leq \sum_{i=1}^n w_i A_i. \quad (1.5)$$

For further information about the matrix power mean, Karcher mean and their properties, we refer the readers to [12, 11, 13] and references therein.

It is well known that for the two positive definite matrices A, B , if $A \geq B$, then

$$A^p \geq B^p \quad (0 \leq p \leq 1). \quad (1.6)$$

In general (1.6) is not true for $p > 1$. Let Φ be a unital positive linear map. The following inequality is known as the Choi inequality see [5, 9].

$$\Phi(A)^{-1} \leq \Phi(A^{-1}). \quad (1.7)$$

Ando [1] proved that if Φ is a positive linear map, then for positive definite matrices $A, B \in \mathfrak{B}(\mathcal{H})$ we have

$$\Phi(A \sharp B) \leq \Phi(A) \sharp \Phi(B). \quad (1.8)$$

A reverse of the Ando's inequality (1.8) is as follows: If $A, B \in \mathcal{M}_n$ and $0 < m \leq A, B \leq M$, Then

$$\Phi(A) \sharp \Phi(B) \leq \frac{M+m}{2\sqrt{Mm}} \Phi(A \sharp B).$$

By inequality (1.6) we get

$$(\Phi(A) \sharp \Phi(B))^p \leq \left(\frac{M+m}{2\sqrt{Mm}} \right)^p \Phi^p(A \sharp B), \quad (0 < p \leq 1). \quad (1.9)$$

Marshall and Olkin [16] proved that a counterpart of Choi's inequality (1.7) as follows

$$\Phi(A^{-1}) \leq \frac{(M+m)^2}{4Mm} \Phi(A)^{-1} \quad (1.10)$$

for positive definite A with $0 < m \leq A \leq M$. In addition Lin [14] and Fu [7] improved inequality (1.10) for $p \geq 2$.

The matrix power means satisfy the following inequality: for each $t \in (0, 1]$

$$\Phi(P_t(\omega; \mathbb{A})) \leq P_t(\omega; \Phi(\mathbb{A})), \quad (1.11)$$

where $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices and $\Phi(\mathbb{A}) = (\Phi(A_1), \dots, \Phi(A_n))$.

Dehghani et al. [6] established counterparts of (1.11) involving matrix power means as following:

$$P_t^2(\omega; \Phi(\mathbb{A})) \leq \left(\frac{(m+M)^2}{4mM} \right)^2 \Phi^2(P_t(\omega; \mathbb{A}))$$

for all $t \in [-1, 1] \setminus \{0\}$ and $0 < m \leq A_i \leq M$.

Using inequality (1.6) we get

$$P_t^p(\omega; \Phi(\mathbb{A})) \leq \left(\frac{(m+M)^2}{4mM} \right)^p \Phi^p(P_t(\omega; \mathbb{A})), \quad (0 < p \leq 2) \quad (1.12)$$

It is interesting to ask whenever the inequality (1.12) is true for $p \geq 2$. This is the first motivation of this paper. moreover, we improve inequality (1.9) for $p \geq 2$. We also obtain some reverses of (1.2). In the last section, we establish several refinements of obtained inequalities.

2. MAIN RESULTS

To prove our first result, we need the following lemmas.

Lemma 2.1. [4, 2, 3, 8] *Let $A, B \in \mathcal{M}_n$ be positive definite matrices and $\alpha > 0$. Then*

- (i) $\|AB\| \leq \frac{1}{4}\|A+B\|^2$.
- (ii) $\|A^\alpha + B^\alpha\| \leq \|(A+B)^\alpha\|$.
- (iii) $A \leq \alpha B$ if and only if $\|A^{\frac{1}{2}}B^{-\frac{1}{2}}\| \leq \alpha^{\frac{1}{2}}$.
- (iv) If $0 \leq A \leq B$ and $0 < m \leq A \leq M$, then $A^2 \leq \frac{(M+m)^2}{4Mm} B^2$.

Lemma 2.2. [10] *Let $A \in \mathcal{M}_n$ be positive definite. Then $A \leq tI$ if and only if $\|A\| \leq t$ if and only if $\begin{bmatrix} tI & A \\ A^* & tI \end{bmatrix}$ is positive.*

Theorem 2.3. *Let $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices with $0 < m \leq A_i \leq M$, ($i = 1, \dots, n$) for some scalars $m < M$ and $\omega = (w_1, \dots, w_n)$ a weight vector. If Φ is a unital positive linear map, then*

$$P_t^p(\omega; \Phi(\mathbb{A})) \leq \left(\frac{(m+M)^2}{4^{\frac{p}{2}}mM} \right)^p \Phi^p(P_t(\omega; \mathbb{A})) \quad (2.1)$$

for every $p \geq 2$ and $t \in [-1, 1] \setminus \{0\}$.

Proof. By Lemma 2.1(iii), inequality (2.1) is equivalent to

$$\left\| P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A})) \Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A})) \right\| \leq \frac{(m+M)^p}{4M^{\frac{p}{2}}m^{\frac{p}{2}}}. \quad (2.2)$$

Hence, it is enough to prove inequality (2.2). So

$$\begin{aligned}
M^{\frac{p}{2}}m^{\frac{p}{2}}\left\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A}))\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\right\| &= \left\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A}))M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\right\| \\
&\leq \frac{1}{4}\left\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A})) + M^{\frac{p}{2}}m^{\frac{p}{2}}\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\right\|^2 \\
&\quad \text{(by Lemma 2.1(i))} \\
&\leq \frac{1}{4}\left\|(P_t(\omega; \Phi(\mathbb{A})) + Mm\Phi^{-1}(P_t(\omega; \mathbb{A})))^{\frac{p}{2}}\right\|^2 \\
&\quad \text{(by Lemma 2.1(ii))} \\
&= \frac{1}{4}\|(P_t(\omega; \Phi(\mathbb{A})) + Mm\Phi^{-1}(P_t(\omega; \mathbb{A})))\|^p \\
&\leq \frac{1}{4}\left\|\sum_{i=1}^n w_i\Phi(A_i) + Mm\Phi(P_t(\omega; \mathbb{A})^{-1})\right\|^p \\
&\quad \text{(by (1.7))} \\
&\leq \frac{1}{4}\left\|\sum_{i=1}^n w_i\Phi(A_i) + Mm\Phi\left(\sum_{i=1}^n w_iA_i^{-1}\right)\right\|^p \\
&\quad \text{(by (1.2))} \\
&= \frac{1}{4}\left\|\sum_{i=1}^n w_i\left(\Phi(A_i) + Mm\Phi(A_i^{-1})\right)\right\|^p. \quad (2.3)
\end{aligned}$$

It follows from $0 < m \leq A_i \leq M$ that $(M - A_i)(m - A_i)A_i^{-1} \leq 0$ ($i = 1, 2, \dots, n$). Hence

$$Mm\Phi(A_i^{-1}) + \Phi(A_i) \leq M + m \quad (i = 1, 2, \dots, n) \quad (2.4)$$

Using inequalities (2.3) and (2.4) we get

$$\|P_t^{\frac{p}{2}}(\omega; \Phi(\mathbb{A}))\Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A}))\| \leq \frac{(m + M)^p}{4M^{\frac{p}{2}}m^{\frac{p}{2}}}.$$

Thus, this completes the proof. \square

In the following result we state that inequality (1.9) is valid for any $p \geq 2$.

Corollary 2.4. *Let $A, B \in \mathcal{M}(\mathbb{C})$ be positive definite matrices such that $0 < m \leq A, B \leq M$ for some scalars $m < M$ and $\alpha \in [0, 1]$. Then*

$$(\Phi(A)\sharp_{\alpha}\Phi(B))^p \leq \left(\frac{(m + M)^2}{4^{\frac{2}{p}}mM}\right)^p \Phi^p(A\sharp_{\alpha}B),$$

for any $p \geq 2$ and unital positive linear map Φ .

Proof. Using this fact $P_t(1 - \alpha, \alpha; A, B) = A \sharp_\alpha B$, ($\alpha \in [0, 1]$) and $n = 2, w_1 = 1 - \alpha$ and $w_2 = \alpha$ in inequality (2.1), we get the desired result. \square

Corollary 2.5. *Let $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices with $0 < m \leq A_i \leq M$, ($i = 1, \dots, n$) for some scalars $m < M$ and $\omega = (w_1, \dots, w_n)$ a weight vector. If Φ is a unital positive linear map, then*

$$\Lambda^p(\omega; \Phi(\mathbb{A})) \leq \left(\frac{(m + M)^2}{4^{\frac{2}{p}} m M} \right)^p \Phi^p(\Lambda(\omega; \mathbb{A}))$$

for every $p \geq 2$ and $t \in [-1, 1] \setminus \{0\}$.

Proof. The proof follows from Theorem 2.3 and relation (1.4). \square

Theorem 2.6. *Let $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices such that $0 < m \leq A_i \leq M$, ($i = 1, \dots, n$) for some scalars $m < M$ and $\omega = (w_1, \dots, w_n)$ a weight vector. Then*

$$\sum_{i=1}^n w_i A_i \leq \frac{(M + m)^2}{4Mm} P_t(\omega; \mathbb{A}), \quad (2.5)$$

where $t \in [-1, 1] \setminus \{0\}$.

Proof. If we put $\Phi(A) = \sum_{i=1}^n w_i A_i$, then for $t \in (0, 1]$ we have

$$\begin{aligned} \sum_{i=1}^n w_i A_i = \Phi(A) &\leq \frac{(M + m)^2}{4Mm} \left(\sum_{i=1}^n w_i A_i^{-1} \right)^{-1} && \text{(by (1.10))} \\ &\leq \frac{(M + m)^2}{4Mm} P_t(\omega; \mathbb{A}) && \text{(by (1.2)).} \end{aligned}$$

Therefore

$$\sum_{i=1}^n w_i A_i \leq \frac{(M + m)^2}{4Mm} P_t(\omega; \mathbb{A}).$$

Inequality (2.5) follows from a similar fashion for $t \in [-1, 0)$. \square

Remark 2.7. As special case for $\mathbb{A} = (A, B)$ and $\omega = (w_1, w_2)$ with $w_1 = w_2 = \frac{1}{2}$, we have the following inequality:

$$\frac{A + B}{2} \leq \frac{(M + m)^2}{4Mm} (A \sharp B),$$

which is counterpart of AM-GM inequality (1.1).

Corollary 2.8. *Let $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices with $0 < m \leq A_i \leq M$, ($i = 1, \dots, n$) for some scalars $m \leq M$ and $\omega = (w_1, \dots, w_n)$ a weight vector. Then*

$$\sum_{i=1}^n w_i A_i \leq \frac{(M+m)^2}{4Mm} \Lambda(\omega; \mathbb{A}), \quad (2.6)$$

where $t \in [-1, 1] \setminus \{0\}$.

Remark 2.9. Inequalities (2.5) and (2.6) can be regarded as a counterpart of inequalities (1.2) and (1.5), respectively. By inequalities (2.5) and (1.11), we can obtain the following operator inequality

$$\begin{aligned} \Phi\left(\sum_{i=1}^n w_i A_i\right) &\leq \frac{(M+m)^2}{4Mm} \Phi(P_t(\omega; \mathbb{A})) \\ &\leq \frac{(M+m)^2}{4Mm} P_t(\omega; \Phi(\mathbb{A})). \end{aligned} \quad (2.7)$$

Now, by applying inequality (1.6) we get

$$\Phi^p\left(\sum_{i=1}^n w_i A_i\right) \leq \left(\frac{(M+m)^2}{4Mm}\right)^p P_t^p(\omega; \Phi(\mathbb{A})) \quad (2.8)$$

for $0 < p \leq 1$.

In the next theorem, we show that inequality (2.8) is valid for $p > 1$.

Theorem 2.10. *Let $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices with $0 < m \leq A_i \leq M$, ($i = 1, \dots, n$) for some scalars $m < M$ and $\omega = (w_1, \dots, w_n)$ a weight vector. Then*

$$\Phi^p\left(\sum_{i=1}^n w_i A_i\right) \leq \alpha^p \Phi^p(P_t(\omega; \mathbb{A})), \quad (2.9)$$

where $t \in [-1, 1] \setminus \{0\}$, $p > 1$ and $\alpha = \max\left\{\frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}} Mm}\right\}$.

Proof. First we show inequality (2.9) for $p = 2$. We have

$$\begin{aligned}
Mm \left\| \Phi \left(\sum_{i=1}^n w_i A_i \right) \Phi^{-1}(P_t(\omega; \mathbb{A})) \right\| &= \left\| \Phi \left(\sum_{i=1}^n w_i A_i \right) Mm \Phi^{-1}(P_t(\omega; \mathbb{A})) \right\| \\
&\leq \frac{1}{4} \left\| \Phi \left(\sum_{i=1}^n w_i A_i \right) + Mm \Phi^{-1}(P_t(\omega; \mathbb{A})) \right\|^2 \\
&\quad \text{(by Lemma 2.1)} \\
&\leq \frac{1}{4} \left\| \Phi \left(\sum_{i=1}^n w_i A_i \right) + Mm \Phi \left(\sum_{i=1}^n w_i A_i^{-1} \right) \right\|^2 \\
&\leq \frac{1}{4} (M + m)^2,
\end{aligned}$$

whence

$$\left\| \Phi \left(\sum_{i=1}^n w_i A_i \right) \Phi^{-1}(P_t(\omega; \mathbb{A})) \right\| \leq \frac{(M + m)^2}{4Mm}.$$

Hence

$$\Phi^2 \left(\sum_{i=1}^n w_i A_i \right) \leq \left(\frac{(M + m)^2}{4Mm} \right)^2 \Phi^2(P_t(\omega; \mathbb{A})).$$

Therefore

$$\Phi^p \left(\sum_{i=1}^n w_i A_i \right) \leq \left(\frac{(M + m)^2}{4Mm} \right)^p \Phi^p(P_t(\omega; \mathbb{A})), \quad (0 \leq p \leq 2) \quad (2.10)$$

Now, we prove inequality (2.9) for $p > 2$. In this case we have

$$\begin{aligned}
&\left\| \Phi^{\frac{p}{2}} \left(\sum_{i=1}^n w_i A_i \right) M^{\frac{p}{2}} m^{\frac{p}{2}} \Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A})) \right\| \\
&\leq \frac{1}{4} \left\| \Phi^{\frac{p}{2}} \left(\sum_{i=1}^n w_i A_i \right) + M^{\frac{p}{2}} m^{\frac{p}{2}} \Phi^{-\frac{p}{2}}(P_t(\omega; \mathbb{A})) \right\|^2 \\
&\quad \text{(by Lemma 2.1(i))} \\
&\leq \frac{1}{4} \left\| \left(\Phi \left(\sum_{i=1}^n w_i A_i \right) + Mm \Phi^{-1}(P_t(\omega; \mathbb{A})) \right)^{\frac{p}{2}} \right\|^2 \\
&\quad \text{(by Lemma 2.1(ii))} \\
&= \frac{1}{4} \left\| \Phi \left(\sum_{i=1}^n w_i A_i \right) + Mm \Phi^{-1}(P_t(\omega; \mathbb{A})) \right\|^p \\
&\leq \frac{(M + m)^p}{4}.
\end{aligned}$$

Hence

$$\left\| \Phi^{\frac{p}{2}} \left(\sum_{i=1}^n w_i A_i \right) \Phi^{-\frac{p}{2}} (P_t(\omega; \mathbb{A})) \right\| \leq \frac{1}{4} \left(\frac{(M+m)^p}{M^{\frac{p}{2}} m^{\frac{p}{2}}} \right).$$

Thus

$$\Phi^p \left(\sum_{i=1}^n w_i A_i \right) \leq \left(\frac{(M+m)^2}{4^{\frac{2}{p}} M m} \right)^p \Phi^p (P_t(\omega; \mathbb{A})). \quad (2.11)$$

Now, if we take $\alpha = \max \left\{ \frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}} M m} \right\}$, then by (2.10) and (2.11) we get the desired result. \square

Remark 2.11. By letting $\mathbb{A} = (A, B)$ and $\omega = (w_1, w_2)$ with $w_1 = w_2 = \frac{1}{2}$ in Theorem 2.10, the following inequalities are hold:

$$\Phi^p \left(\frac{A+B}{2} \right) \leq \alpha^p \Phi^p (A \sharp B),$$

Which appeared in [9, Theorem 4]. where $\alpha = \max \left\{ \frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}} M m} \right\}$.

Corollary 2.12. Let $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices with $0 < m \leq A_i \leq M$, ($i = 1, \dots, n$) for some scalars $m \leq M$ and $\omega = (w_1, \dots, w_n)$ a weight vector, and let $t \in [-1, 1] \setminus \{0\}$. Then

$$\Phi^p \left(\sum_{i=1}^n w_i A_i \right) \leq \alpha^p \Phi^p \Lambda(\omega; \mathbb{A}),$$

where $p \geq 1$ and $\alpha = \max \left\{ \frac{(M+m)^2}{4Mm}, \frac{(M+m)^2}{4^{\frac{2}{p}} M m} \right\}$.

In the next result we extend inequalities (2.1) and (2.9) to the follwing form.

Theorem 2.13. Let $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices with $0 < m \leq A_i \leq M$, ($i = 1, \dots, n$) for some scalars $m \leq M$ and $\omega = (w_1, \dots, w_n)$ a weight vector, let $t \in [-1, 1] \setminus \{0\}$ and Φ be a positive unital linear map. Then

$$P_t^p(\omega; \mathbb{A}) \Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \mathbb{A})) P_t^p(\omega; \mathbb{A}) \leq 2\alpha^p$$

and

$$\Phi^p \left(\sum_{i=1}^n w_i A_i \right) \Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \mathbb{A})) \Phi^p \left(\sum_{i=1}^n w_i A_i \right) \leq 2\alpha^p, \quad (2.12)$$

where $p > 0$ and $\alpha = \max \left\{ \frac{(m+M)^2}{4mM}, \frac{(m+M)^2}{4^{\frac{1}{p}} m M} \right\}$.

Proof. By inequality (1.12) and Lemma 2.1(iii) for $0 < p \leq 1$ we have

$$||P_t^p(\omega; \mathbb{A})\Phi^{-p}(P_t(\omega; \mathbb{A}))|| \leq \left(\frac{(m+M)^2}{4mM}\right)^p.$$

We put $\alpha = \frac{(m+M)^2}{4mM}$. Using Lemma 2.2 we get

$$\begin{bmatrix} \alpha^p I & P_t^p(\omega; \mathbb{A})\Phi^{-p}(P_t(\omega; \mathbb{A})) \\ \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \mathbb{A}) & \alpha^p I \end{bmatrix}$$

and

$$\begin{bmatrix} \alpha^p I & \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \mathbb{A}) \\ P_t^p(\omega; \mathbb{A})\Phi^{-p}(P_t(\omega; \mathbb{A})) & \alpha^p I \end{bmatrix}$$

are positive. Hence

$$\begin{bmatrix} 2\alpha^p I & P_t^p(\omega; \mathbb{A})\Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \mathbb{A}) \\ \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \mathbb{A}) + P_t^p(\omega; \mathbb{A})\Phi^{-p}(P_t(\omega; \mathbb{A})) & 2\alpha^p I \end{bmatrix}$$

is positive. Using Lemma 2.2 we get

$$P_t^p(\omega; \mathbb{A})\Phi^{-p}(P_t(\omega; \mathbb{A})) + \Phi^{-p}(P_t(\omega; \mathbb{A}))P_t^p(\omega; \mathbb{A}) \leq 2\alpha^p.$$

For $p > 1$, using inequality (2.1) with the same argument, we get the desired inequality. Inequality (2.13) is proved by using Theorem 2.10 and a similar method. \square

Corollary 2.14. *Let $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices with $0 < m \leq A_i \leq M$, ($i = 1, \dots, n$) for some scalars $m \leq M$ and $\omega = (w_1, \dots, w_n)$ a weight vector, Φ be a positive unital linear map. Then*

$$\Lambda^p(\omega; \mathbb{A})\Phi^{-p}(\Lambda(\omega; \mathbb{A})) + \Phi^{-p}(\Lambda(\omega; \mathbb{A}))\Lambda^p(\omega; \mathbb{A}) \leq 2\alpha^p$$

and

$$\Phi^p\left(\sum_{i=1}^n w_i A_i\right)\Phi^{-p}(\Lambda(\omega; \mathbb{A})) + \Phi^{-p}(\Lambda(\omega; \mathbb{A}))\Phi^p\left(\sum_{i=1}^n w_i A_i\right) \leq 2\alpha^p, \quad (2.13)$$

where $p > 0$ and $\alpha = \max\left\{\frac{(m+M)^2}{4mM}, \frac{(m+M)^2}{4^{\frac{1}{p}}mM}\right\}$.

In the next result, we would like to obtain unitary invariant norm inequality involving matrix power means.

Proposition 2.15. *Let $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices with $0 < m \leq A_i \leq M$, ($i = 1, \dots, n$) for some scalars $m \leq M$ and $\omega = (w_1, \dots, w_n)$ a weight vector, and let $|||\cdot|||$ be an unitary invariant norm. Then for $t \in (0, 1]$*

$$|||P_t(\omega; \mathbb{A})||| \leq \sum_{i=1}^n w_i |||A_i||| \quad \text{and} \quad |||P_{-t}(\omega; \mathbb{A})||| \geq \left(\sum_{i=1}^n w_i |||A_i^{-1}|||\right)^{-1}.$$

Proof. Let $X = P_t(\omega; \mathbb{A})$. Then

$$\begin{aligned} |||X||| &= |||P_t(\omega; \mathbb{A})||| \leq \sum_{i=1}^n w_i |||X \sharp_t A_i||| \\ &\leq \sum_{i=1}^n w_i |||(1-t)X + tA_i||| \\ &\leq |||(1-t)X||| \sum_{i=1}^n w_i + t \sum_{i=1}^n w_i |||A_i|||, \end{aligned}$$

which implies that $|||P_t(\omega; \mathbb{A})||| \leq \sum_{i=1}^n w_i |||A_i|||$. For second inequality, it follows from $|||A^{-1}||| \geq |||A|||^{-1}$ for any $A > 0$ that

$$|||P_{-t}(\omega; \mathbb{A})||| = |||P_t(\omega; \mathbb{A}^{-1})^{-1}||| \geq |||P_t(\omega; \mathbb{A}^{-1})|||^{-1} \geq \left(\sum_{i=1}^n w_i |||A_i^{-1}||| \right)^{-1}.$$

□

3. SOME REFINEMENTS

In this section, we give a refinement of inequality (2.9). This inequality can be refined by a similar method that known in [17].

Theorem 3.1. *Let $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices with $0 < m \leq A_i \leq M$, ($i = 1, \dots, n$) for some scalars $m \leq M$ and $\omega = (w_1, \dots, w_n)$ a weight vector, and let $t \in [-1, 1] \setminus \{0\}$. Then for every positive unital linear map Φ*

$$\Phi^{2p} \left(\sum_{i=1}^n w_i A_i \right) \leq \frac{(K(M^2 + m^2))^{2p}}{16M^{2p}m^{2p}} \Phi^{2p}(P_t(\omega; \mathbb{A})), \quad (3.1)$$

where $p \geq 2$ and $K = \frac{(M+m)^2}{4mM}$.

Proof. For $p \geq 2$, we have

$$\begin{aligned}
& \left\| \Phi^p \left(\sum_{i=1}^n w_i A_i \right) M^p m^p \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\| \\
& \leq \frac{1}{4} \left\| K^{\frac{p}{2}} \Phi^p \left(\sum_{i=1}^n w_i A_i \right) + \left(\frac{M^2 m^2}{K} \right)^{\frac{p}{2}} \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\|^2 \\
& \quad \text{(by Lemma 2.1(i))} \\
& \leq \frac{1}{4} \left\| \left(K \Phi^2 \left(\sum_{i=1}^n w_i A_i \right) + \frac{M^2 m^2}{K} \Phi^{-2}(P_t(\omega; \mathbb{A})) \right)^{\frac{p}{2}} \right\|^2 \\
& \quad \text{(by Lemma 2.1(ii))} \\
& = \frac{1}{4} \left\| \left(K \Phi^2 \left(\sum_{i=1}^n w_i A_i \right) + \frac{M^2 m^2}{K} \Phi^{-2}(P_t(\omega; \mathbb{A})) \right) \right\|^p \\
& \leq \frac{1}{4} \left\| \left(K \Phi^2 \left(\sum_{i=1}^n w_i A_i \right) + \frac{M^2 m^2}{K} \Phi^2(P_t(\omega; \mathbb{A})^{-1}) \right) \right\|^p \\
& \quad \text{(by (1.7))} \\
& \leq \frac{1}{4} \left\| K \Phi^2 \left(\sum_{i=1}^n w_i A_i \right) + K M^2 m^2 \Phi^2 \left(\sum_{i=1}^n w_i A_i^{-1} \right) \right\|^p \\
& \quad \text{(by Lemma 2.1(iv))} \\
& = \frac{1}{4} (K(M^2 + m^2))^p \quad \text{(by [15, 4.7]).}
\end{aligned}$$

Hence

$$\left\| \Phi^p \left(\sum_{i=1}^n w_i A_i \right) \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\| \leq \frac{1}{4} \left(\frac{K(M^2 + m^2)}{Mm} \right)^p. \quad (3.2)$$

Since (3.2) is equivalent to (3.1), so inequality (3.1) holds. \square

Remark 3.2. If we put $\mathbb{A} = (A, B)$ and $\omega = (w_1, w_2)$ with $w_1 = w_2 = \frac{1}{2}$ in Theorem 3.1, then we get [17, Theorem 2.6] as follows:

$$\Phi^{2p} \left(\frac{A+B}{2} \right) \leq \frac{(K(M^2 + m^2))^{2p}}{16M^2 m^2} \Phi^{2p}(A \sharp B).$$

Theorem 3.3. Let $\mathbb{A} = (A_1, \dots, A_n)$ be a n -tuple of positive definite matrices with $0 < m \leq A_i \leq M$, ($i = 1, \dots, n$) for some scalars $m \leq M$ and $\omega = (w_1, \dots, w_n)$ a weight vector, and let $t \in [-1, 1] \setminus \{0\}$. Then for every positive unital linear map Φ

$$P_t^{2p}(\omega; \Phi(\mathbb{A})) \leq \frac{(K(M^2 + m^2))^{2p}}{16M^{2p} m^{2p}} \Phi^{2p}(P_t(\omega; \mathbb{A})), \quad (3.3)$$

where $p \geq 2$ and $K = \frac{(M+m)^2}{4mM}$.

Proof. For $p \geq 2$, we have

$$\begin{aligned}
& \left\| P_t^p(\omega; \Phi(\mathbb{A})) M^p m^p \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\| \\
& \leq \frac{1}{4} \left\| P_t^p(\omega; \Phi(\mathbb{A})) + (M^2 m^2)^{\frac{p}{2}} \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\|^2 \\
& \quad \text{(by Lemma 2.1(i))} \\
& \leq \frac{1}{4} \left\| \left(P_t^2(\omega; \Phi(\mathbb{A})) + M^2 m^2 \Phi^{-2}(P_t(\omega; \mathbb{A})) \right)^{\frac{p}{2}} \right\|^2 \\
& \quad \text{(by Lemma 2.1(ii))} \\
& = \frac{1}{4} \left\| \left(P_t^2(\omega; \Phi(\mathbb{A})) + M^2 m^2 \Phi^{-2}(P_t(\omega; \mathbb{A})) \right) \right\|^p \\
& \leq \frac{1}{4} \left\| \left(P_t^2(\omega; \Phi(\mathbb{A})) + M^2 m^2 \Phi^2(P_t^{-1}(\omega; \mathbb{A})) \right) \right\|^p \\
& \quad \text{(by (1.7))} \\
& \leq \frac{1}{4} \left\| K \left(\sum_{i=1}^n w_i \Phi(A_i) \right)^2 + M^2 m^2 K \Phi^2 \left(\sum_{i=1}^n w_i A_i^{-1} \right) \right\|^p \\
& \quad \text{(by Lemma 2.1(iv))} \\
& = \frac{1}{4} \left\| K \Phi^2 \left(\sum_{i=1}^n w_i A_i \right) + M^2 m^2 K \left(\sum_{i=1}^n w_i A_i^{-1} \right) \right\|^p \\
& = \frac{1}{4} (K(M^2 + m^2))^p \quad \text{(by [15, 4.7])}.
\end{aligned}$$

Therefore

$$\left\| P_t^p(\omega; \Phi(\mathbb{A})) \Phi^{-p}(P_t(\omega; \mathbb{A})) \right\| \leq \frac{1}{4} \left(\frac{K(M^2 + m^2)}{Mm} \right)^p.$$

Since the last inequality is equivalent to (3.3), thus this completes the proof. \square

Remark 3.4. As special case for $\mathbb{A} = (A, B)$ and $\omega = (w_1, w_2)$ with $w_1 = w_2 = \frac{1}{2}$, Theorem 3.3 is a refinement of Corollary 2.4.

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